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Ultraspherical-polynomials approximation to the radiative heat transfer in a slab with reflective boundaries

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Abstract

The ultraspherical-polynomials approximation or the $P_N^{(\lambda)}$ method, which is obtained by incorporating all approximations that employ ultraspherical polynomials into a unified form, is applied to the radiative transfer problem in plane-parallel, absorbing, emitting, non-isothermal, gray medium with linearly anisotropic scattering. The unique $P_N^{(\lambda)}$ formulation provides a simple means to make comparative assessments and to analyze some qualitative aspects of various ultraspherical-polynomials approximations. Effects of the order of approximation, optical thickness, specular reflection, anisotropic scattering, and change of the source term on results are investigated for different pre-selected values of λ, each leading to a different approximation. All results obtained by the $P_N^{(\lambda)}$ method are consistent in themselves, equiconvergent, and in good agreement with the comparable data in literature and with the results obtained from the computational fluid dynamics code FLUENT.

Keywords: Radiative transfer; Ultraspherical polynomials; Anisotropic scattering

1. Introduction

Many methods have been proposed and successfully applied to a large variety of transfer/transport problems: spherical-harmonics method [1–7], Case's normal-mode expansion technique [3,8,9], solution with Chandrasekhar's X and Y functions [10], discrete-ordinates methods [1,2], Monte Carlo methods [11], zonal method [12] and Chebyshev-polynomials approximation [13,14]. Among them, Legendre polynomials (in spherical harmonics) and, to a lesser extent, Chebyshev polynomials of the first kind have been widely used for analytical treatment of the angular dependency of the flux (or intensity) in order to obtain approximate solutions to the transport equation.

The spherical-harmonics (P_N) method was first proposed as an eigenvalue problem by Davison [7] and has found profound applications in neutron transport. The approximation using the Chebyshev polynomials of the first kind (T_N) was originally applied by Aspelund [13] and Conkie [14] to the neutron transport equation in slab geometry.

In a recent study by Yilmazer and Tombakoglu [15], all ultraspherical-polynomials approximations have been incorporated into a single formulation to obtain eigenspectrum of the transport operator, which was then applied to slab-criticality problems of neutron transport.

The principal idea is that Legendre and Chebyshev polynomials belong to a more general class of one-variable classical polynomials known as ultraspherical polynomials, denoted by $P_N^{(\lambda)}$. Then, all approximations employing ultraspherical polynomials can be unified as "ultraspherical-polynomials approximation", or $P_N^{(\lambda)}$ method. Each value of λ leads to a different approximation, including the spherical harmonics when $\lambda = 1/2$ [$P_N = P_N^{(1/2)}$], Chebyshev polynomials of the first kind when $\lambda = 0$ [$T_N = P_N^{(0)}$], and Chebyshev polynomials of the second kind when $\lambda = 1$ [$U_N = P_N^{(1)}$].

This article focuses on applying that technique to the radiative transfer equation in plane-parallel, absorbing, emitting,

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Nomenclature

B	linear anisotropy coefficient
g_m	eigenfunction
h_m^λ	normalization constant of ultraspherical polynomials
I	intensity of the radiation . . . . $\mathrm{W\,m^{-2}\,sr^{-1}}$
N	order of approximation
p	phase function
P_m	Legendre polynomial of order m
$P_m^{(\lambda)}$	ultraspherical polynomial of order m with variable λ
q	radiative heat flux . . . . $\mathrm{W\,m^{-2}}$
Q	heat flux function
S	source term
T	temperature . . . . K
T_m	Chebyshev polynomial of the first kind of order m
U_m	Chebyshev polynomial of the second kind of order m

Greek symbols

α	strongly forward scattering probability
β	strongly backward scattering probability
δ	Dirac delta function
δ_{mn}	Kronecker delta
ε	emissivity
λ	variable of ultraspherical polynomials
μ	scattering cosine
ν	eigenvalue
ω	single scattering albedo
ϕ_m	the mth ultraspherical moment of the intensity
ρ	reflectivity
τ	optical variable
σ	Stefan–Boltzmann constant . . . . $\mathrm{W\,m^{-2}\,K^{-4}}$

Subscripts, superscripts

0	optical dimension
1	left boundary
2	right boundary
b	blackbody
d	diffuse
h	homogeneous
p	particular
s	specular
$*$	heat flux function obtained using principle of reflection
$\bar{}$	linearly varying source
$\hat{}$	exponentially varying source

non-isothermal gray medium with linearly anisotropic scattering, and specularly and diffuse reflecting boundaries.

The same problem was attacked by Beach et al. [16], who used Case's normal-mode expansion technique and produced benchmark-quality results. In a recent study, Atalay obtained numerical solutions to the problem by the spherical-harmonics method [17]. The results of Beach et al. [16] are to be used as the main basis for comparisons. The spherical-harmonics results of Atalay [17] are expected to be reproduced as a special case of the $P_N^{(\lambda)}$ method for $\lambda = 1/2$. For exponentially varying source terms, results from the CFD code FLUENT are also to be obtained and included in comparative assessments.

Some qualitative aspects (mainly related to convergence characteristics) of different ultraspherical-polynomials approximations are expected to be observed. Effects of the order of approximation, optical thickness, specular reflection, anisotropic scattering, and change of the source term on results are to be analyzed for different pre-selected values of λ. It is anticipated that the unified formulation provides a simple means to find out which special approximation (that is, which value of λ) is better for a particular set of conditions and parameters of the problem.

In summary, this study is an attempt to unify all ultraspherical-polynomials approximations by a single formulation for obtaining solutions to the radiative transfer problem in slab geometry. First, the $P_N^{(\lambda)}$ formulation is to be put forward, then, numerical results are to be produced, and finally, comparative assessments are to be made.

2. The equation of radiative transfer and boundary conditions

The equation of radiative transfer for a plane-parallel, gray medium with azimuthal symmetry is given as [3]

$$\mu\frac{\partial I(\tau,\mu)}{\partial\tau} + I(\tau,\mu) = (1-\omega)I_b\big(T(\tau)\big) + \frac{\omega}{2}\int_{-1}^{1} p(\mu,\mu')I(\tau,\mu')\,\mathrm{d}\mu' \tag{1}$$

The boundary conditions are

$$I(0,\mu) = \varepsilon_1\frac{\sigma}{\pi}T_1^4 + \rho_1^s I(0,-\mu) + 2\rho_1^d\int_0^1 I(0,-\mu')\mu'\,\mathrm{d}\mu' \qquad \mu\in[0,1] \tag{2}$$

$$I(\tau_0,-\mu) = \varepsilon_2\frac{\sigma}{\pi}T_2^4 + \rho_2^s I(\tau_0,\mu) + 2\rho_2^d\int_0^1 I(\tau_0,\mu')\mu'\,\mathrm{d}\mu' \qquad \mu\in[0,1] \tag{3}$$

In the equations above,

$I(\tau,\mu)$: radiative intensity,
$p(\mu,\mu')$: phase function or scattering kernel,
τ: optical variable; $\tau\in[0,\tau_0]$,
μ: direction cosine measured from the positive τ-axis; $\mu\in[-1,1]$,
ω: single scattering albedo (ratio of the scattering coefficient to the extinction coefficient),

$I_b[T(\tau)]$: prescribed frequency-integrated Planck function,
σ: Stefan–Boltzmann constant,
ε: emissivity,
ρ: reflectivity; $\rho_i = \rho_i^d + \rho_i^s$, where d and s denote "diffuse" and "specular", and i denotes surfaces 1 and 2.

The boundary surfaces 1 and 2 are positioned at $\tau = 0$ and $\tau = \tau_0$, respectively and kept at uniform temperatures T_1 and T_2. The surfaces are diffuse emitters with emissivities ε_1 and ε_2.

Eq. (1) can be simplified to

$$\mu \frac{\partial I(\tau,\mu)}{\partial \tau} + I(\tau,\mu) = S(\tau) + \frac{\omega}{2}\int_{-1}^{1} p(\mu,\mu') I(\tau,\mu')\,\mathrm{d}\mu' \quad (4)$$

where

$$a_i = \varepsilon_i(\sigma/\pi)T_i^4, \quad b_i = \rho_i^s, \quad d_i = 2\rho_i^d, \quad i = 1 \text{ and } 2$$

$$S(\tau) = \frac{(1-\omega)}{\pi}\sigma T^4(\tau) \quad (5)$$

The boundary conditions become

$$I(0,\mu) = a_1 + b_1 I(0,-\mu) + d_1 \int_0^1 I(0,-\mu')\mu'\,\mathrm{d}\mu' \qquad \mu \in [0,1] \quad (6)$$

$$I(\tau_0,-\mu) = a_2 + b_2 I(\tau_0,\mu) + d_2 \int_0^1 I(\tau_0,\mu')\mu'\,\mathrm{d}\mu' \qquad \mu \in [0,1] \quad (7)$$

3. Solution by ultraspherical-polynomials ($P_n^{(\lambda)}$) approximation

The solution of Eq. (4) is assumed to be

$$I(\tau,\mu) = I_h(\tau,\mu) + I_p(\tau,\mu) \quad (8)$$

where $I_h(\tau,\mu)$ is the solution of the homogeneous problem, Eq. (4) without the source term, and $I_p(\tau,\mu)$ is a particular solution of Eq. (4).

The homogeneous equation is

$$\mu \frac{\partial I_h(\tau,\mu)}{\partial \tau} + I_h(\tau,\mu) = \frac{\omega}{2}\int_{-1}^{1} p(\mu,\mu') I_h(\tau,\mu')\,\mathrm{d}\mu' \quad (9)$$

By ultraspherical-polynomials ($P_n^{(\lambda)}$) approximation, the solution of Eq. (9) is proposed to be

$$I_h(\tau,\mu) = \sum_{n=0}^{\infty} \left(h_n^\lambda\right)^{-1}\left(1-\mu^2\right)^{\lambda-1/2} P_n^{(\lambda)}(\mu)\phi_n(\tau) \quad (10)$$

where $P_n^{(\lambda)}$ is the ultraspherical polynomial of order n and variable λ, $\phi_n(\tau)$ is the nth-order ultraspherical moment of angular flux, and h_n^λ is the normalization constant given by Eq. (11) [18],

$$h_n^\lambda = \frac{2^{1-2\lambda}\pi}{[\Gamma(\lambda)]^2}\frac{\Gamma(n+2\lambda)}{(n+\lambda)\Gamma(n+1)}, \quad \lambda > -1/2,\ \lambda \neq 0 \quad (11)$$

The phase function, or the scattering kernel for linearly anisotropic and strongly peaked backward and/or forward scattering, is in the form,

$$p(\mu,\mu') = \frac{1}{2}(1-\alpha-\beta)(1+B\mu\mu') + \alpha\delta(\mu-\mu') + \beta\delta(\mu+\mu') \quad (12)$$

where δ is the Dirac's delta function, B is the linear anisotropy coefficient, α and β are forward and backward scattering probabilities in a collision, respectively [15]. Note that, for linearly anisotropic scattering, taking $\alpha = \beta = 0$ and scaling by a factor of 2 in order to conform to the conventional form of the radiative transfer equation, Eq. (12) reduces to Eq. (13).

$$p(\mu,\mu') = 1 + B\mu\mu' \quad (13)$$

Introducing the expansion given by Eq. (10) and the phase function given by Eq. (13) into Eq. (9), and taking ultraspherical moment of both sides [15], the recurrence relation for three consecutive moments of the angular flux are obtained as

$$\begin{aligned}
&(m+2\lambda-1)\frac{\mathrm{d}\phi_{m-1}(\tau)}{\mathrm{d}x} + (m+1)\frac{\mathrm{d}\phi_{m+1}(\tau)}{\mathrm{d}x} \\
&\quad + 2(m+\lambda)\left[1-\omega\left(\alpha+(-1)^m\beta\right)\right]\phi_m(\tau) \\
&= \omega(m+\lambda)(1-\alpha-\beta) \\
&\quad \times \left[\phi_0(\tau)\left(1+(-1)^m\right)\frac{\Gamma(m+2\lambda-1)}{\Gamma(2\lambda-1)\Gamma(m+2)} + \frac{B\phi_1(\tau)}{2\lambda}\right. \\
&\quad \left. \times \sum_{k=0}^{[m/2]} \frac{(-1)^k(\lambda)_{m-k}2^{m-2k}}{k!(m-2k)!(m-2k+2)}\left(1-(-1)^m\right)\right]
\end{aligned} \quad (14)$$

where Γ is the Gamma function.

Notice that $\lambda = 1/2$ corresponds to the spherical harmonics (P_N) approximation, and $\lambda = 1$ to the Chebyshev polynomial of the second kind (U_N) approximation. For $\lambda = 0$, special attention is required since $P_m^{(\lambda)}$ cannot be reduced to the Chebyshev polynomial of the first kind (T_m). The relationship between $P_m^{(\lambda)}$ and T_m is given by the limiting condition

$$\lim_{\lambda\to 0} \lambda^{-1} P_m^{(\lambda)}(\mu) = (2/m)T_m(\mu), \quad m \geqslant 1 \quad (15)$$

The normalization constant, defined by Eq. (11), should also be modified accordingly [18].

Using Eq. (15) and following the steps in obtaining Eq. (14), the recurrence relation for $\lambda = 0$ (T_N approximation) is obtained to be

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left[(1+\delta_{m0})\phi_{m+1}(\tau) + \phi_{m-1}(\tau)\right] \\
&\quad + \left(1-\omega(\alpha+(-1)^m\beta)\right)\phi_m(\tau) \\
&= -\omega(1-\alpha-\beta)\left[\frac{(1+(-1)^m)}{2(m^2-1)}\phi_0(\tau)\right. \\
&\quad \left. - B\frac{(1+(-1)^{m+1})}{2(m^2-4)}\phi_1(\tau)\right]
\end{aligned} \quad (16)$$

Eq. (16), while representing the exact case for $\lambda = 0$, gives rise to extra algebraic operations and hampers our unification attempt. Instead of using Eq. (16) for $\lambda = 0$ case, by picking λ small enough (e.g., $\lambda = 10^{-6}$), it becomes possible to use Eq. (14) without any modification. Then, for most practical computational purposes, the recurrence relation given by Eq. (14) describes all ultraspherical-polynomials approximations: $\lambda = 1/2$, exactly corresponding to P_N (spherical harmonics); $\lambda = 1$, exactly corresponding to U_N (Chebyshev of the second kind); and $\lambda =$ small enough, approximately corresponding to T_N (Chebyshev of the first kind).

Eq. (14) constitutes an infinite set of coupled ordinary differential equations whose exact solution is impossible. Neglecting ϕ_{N+1} term ($P_N^{(\lambda)}$ approximation), the resulting set of equations can approximately be solved by the well known procedure [7] for obtaining the eigenspectrum; that is, seeking a solution of the form

$$\phi_m(\tau) = g_m(\nu)\mathrm{e}^{\tau/\nu} \quad (m = 0, 1, \ldots, N) \tag{17}$$

Substitution of Eq. (17) into Eq. (14) results in

$$\begin{aligned}&(m+2\lambda-1)g_{m-1} + (m+1)g_{m+1} \\ &\quad + 2(m+\lambda)\nu\big[1 - \omega\big(\alpha + (-1)^m\beta\big)\big]g_m \\ &= \omega(m+\lambda)(1-\alpha-\beta)\nu \\ &\quad \times \bigg[g_0\big(1+(-1)^m\big)\frac{\Gamma(m+2\lambda-1)}{\Gamma(2\lambda-1)\Gamma(m+2)} \\ &\quad + \frac{Bg_1}{2\lambda}\sum_{k=0}^{[m/2]}\frac{(-1)^k(\lambda)_{m-k}2^{m-2k}}{k!(m-2k)!(m-2k+2)}\big(1-(-1)^m\big)\bigg] \\ &(m = 0, 1, \ldots, N)\end{aligned} \tag{18}$$

Eq. (18) can be solved for g_m recursively with the initial values $g_{-1} = 0$ and $g_0 = 1$. Compatibility of Eqs. (15) is assured if the determinant of coefficients vanishes, which is equivalent to using the well-known closure relationship [7],

$$g_{N+1}(\nu) = 0 \tag{19}$$

Discrete eigenspectrum of the transport operator is formed through finding $N+1$ roots of $g_{N+1}(\nu)$ from Eq. (19). In the $P_N^{(\lambda)}$ approximation, with N odd, there are $N+1$ roots ν_j and $g_n(-\nu_j) = (-1)^n g_n(\nu_j)$. Then, the solution of the radiative transfer equation can be written as

$$\begin{aligned}I(\tau,\mu) &= \sum_{n=0}^{N}\big(h_n^\lambda\big)^{-1}\big(1-\mu^2\big)^{\lambda-1/2}P_n^{(\lambda)}(\mu)\sum_{j=1}^{(N+1)/2}g_n(\nu_j) \\ &\quad \times \big[A_j\mathrm{e}^{-\tau/\nu_j} + (-1)^n B_j\mathrm{e}^{\tau/\nu_j}\big] + I_p(\tau,\mu)\end{aligned} \tag{20}$$

Constants $\{A_j\}$ and $\{B_j\}$ can be determined by inserting Eq. (20) into the boundary conditions, Eqs. (6) and (7), and using the Marshak projection scheme [7]; that is, multiplying by $P_{2i-1}^{(\lambda)}(\mu)$ and integrating over μ from zero to unity, for $i = 1, \ldots, (N+1)/2$. The resulting equations are

$$\begin{aligned}&\sum_{n=0}^{N}\big(h_n^\lambda\big)^{-1}\bigg[\big(1-(-1)^n b_1\big)S_{n,2i-1} - \frac{(-1)^n}{2\lambda}d_1V_{2i-1}S_{n,1}\bigg] \\ &\quad \times \sum_{j=1}^{(N+1)/2}g_n(\nu_j)\big[A_j + (-1)^n B_j\big] = M_i\end{aligned} \tag{21}$$

$$\begin{aligned}&\sum_{n=0}^{N}\big(h_n^\lambda\big)^{-1}\bigg[\big((-1)^n - b_2\big)S_{n,2i-1} - \frac{1}{2\lambda}d_2V_{2i-1}S_{n,1}\bigg] \\ &\quad \times \sum_{j=1}^{(N+1)/2}g_n(\nu_j)\big[A_j\mathrm{e}^{-\tau_0/\nu_j} + (-1)^n B_j\mathrm{e}^{\tau_0/\nu_j}\big] = N_i\end{aligned} \tag{22}$$

Parameters appearing in Eqs. (21) and (22) are defined as follows:

$$S_{n,m} = \int_0^1\big(1-\mu^2\big)^{\lambda-1/2}P_n^\lambda(\mu)P_m^\lambda(\mu)\,\mathrm{d}\mu \tag{23}$$

$$M_i = V_{2i-1}\big[a_1 + d_1Y_1^p\big] + b_1Y_5^p - Y_3^p \tag{24}$$

$$N_i = V_{2i-1}\big[a_2 + d_2Y_2^p\big] + b_2Y_6^p - Y_4^p \tag{25}$$

$$\begin{aligned}V_{2i-1} &= \int_0^1 P_{2i-1}^{(\lambda)}(\mu)\,\mathrm{d}\mu = \bigg(-\frac{4^i\sqrt{\pi}\,\Gamma(\lambda+i-1)}{\Gamma(1/2-i)\Gamma(\lambda)} \\ &\quad + \frac{2\Gamma[2(\lambda+i-1)]}{\Gamma(2\lambda-1)}\bigg)\bigg/2\Gamma(2i+1)\end{aligned} \tag{26}$$

$$\begin{aligned}Y_1^p &= \int_0^1 \mu I_p(0,-\mu)\,\mathrm{d}\mu, \qquad Y_2^p = \int_0^1 \mu I_p(\tau_0,\mu)\,\mathrm{d}\mu \\ Y_3^p &= \int_0^1 P_{2i-1}^{(\lambda)}(\mu)I_p(0,\mu)\,\mathrm{d}\mu \\ Y_4^p &= \int_0^1 P_{2i-1}^{(\lambda)}(\mu)I_p(\tau_0,-\mu)\,\mathrm{d}\mu \\ Y_5^p &= \int_0^1 P_{2i-1}^{(\lambda)}(\mu)I_p(0,-\mu)\,\mathrm{d}\mu \\ Y_6^p &= \int_0^1 P_{2i-1}^{(\lambda)}(\mu)I_p(\tau_0,\mu)\,\mathrm{d}\mu\end{aligned} \tag{27}$$

The parameters can be determined starting with Eq. (27), with a known I_p, and proceeding backward through Eq. (23) for $i = 1, \ldots, (N+1)/2$. The half-range integral defined by Eq. (23) can be expressed as [15]

$$S_{n,m} = \begin{cases} a_p^\lambda\displaystyle\sum_{k=0}^{p-1}(-1)^{k+1}2^k(\lambda)_k\big[a_{p-k-1}^{\lambda+k+1}\big]^{-1}P_{q-k}^{(\lambda+k)}(0) \\ \quad \times P_{p-k-1}^{(\lambda+k+1)}(0) + (-1)^p2^{p+1}(\lambda)_pa_p^\lambda \\ \quad \times \dfrac{(\lambda+p)}{(q-p)(q+p+2\lambda)}P_{q-p-1}^{(\lambda+p+1)}(0) \\ \quad m \neq n,\ p = \mathrm{Min}(m,n),\ q = \mathrm{Max}(m,n) \\ h_n^\lambda/2; \quad m = n\end{cases} \tag{28}$$

The coefficients a_n^λ are given by

$$a_n^\lambda = \frac{(-1)^n\Gamma(\lambda+1/2)\Gamma(n+2\lambda)}{n!2^n\Gamma(2\lambda)\Gamma(n+\lambda+1/2)} \tag{29}$$

Values of $P_n^{(\lambda)}(\mu)$ at $\mu = 0$ are obtained from (http://functions.wolfram.com/05.09.03.0001.01)

$$P_n^{(\lambda)}(0) = \frac{2^n \sqrt{\pi}\,\Gamma(\lambda + n/2)}{\Gamma(\frac{1-n}{2})\Gamma(n+1)\Gamma(\lambda)} \tag{30}$$

Eqs. (21) and (22) constitute a $2(N+1) \times 2(N+1)$ linear system to evaluate $2(N+1)$ unknown coefficients $\{A_j\}$ and $\{B_j\}$, $j = 1, \ldots, N+1$. Solving this system concludes the $P_N^{(\lambda)}$ approximation to the radiative transfer equation in an absorbing, emitting, non-isothermal gray medium between two parallel reflecting boundaries with strongly anisotropic scattering for the non-conservative case ($\omega \neq 1$).

For the conservative case ($\omega = 1$), the $P_N^{(\lambda)}$ approximation, expressed by Eq. (20), requires some modification in order to remove singularities encountered at low order terms. The reader can consult Atalay's work [17] for such a modification in the spherical harmonics (P_N) method. Since adaptation of Atalay's idea to the $P_N^{(\lambda)}$ approximation for the conservative case results in tedious algebraic operations, in this study, it has been preferred to use a value of albedo ω very very close to unity. As long as attention is paid to the precision in numerical work, such an approach, by reducing computational efforts, serves better for the purposes of this study.

4. Radiative heat flux, source terms and particular solutions

The radiative heat flux is expressed by

$$q(\tau) = 2\pi \int_{-1}^{1} \mu I(\tau, \mu)\,\mathrm{d}\mu$$
$$= 2\pi \int_{-1}^{1} \mu\big[I_h(\tau, \mu) + I_p(\tau, \mu)\big]\,\mathrm{d}\mu \tag{31}$$

Using the relation $\mu = \frac{1}{2\lambda} P_1^{\lambda}(\mu)$ [18], the particular and homogeneous parts of the heat flux can be written as

$$q_p(\tau) = 2\pi \int_{-1}^{1} \mu I_p(\tau, \mu)\,\mathrm{d}\mu \tag{32}$$

$$q_h(\tau) = 2\pi \int_{-1}^{1} \mu I_h(\tau, \mu)\,\mathrm{d}\mu \tag{33}$$

Inserting our $P_N^{(\lambda)}$ approximation for the intensity, the first term in Eq. (20), into Eq. (33), $q_h(\tau)$ becomes

$$q_h(\tau) = \frac{\pi}{\lambda} \int_{-1}^{1} P_1^{(\lambda)}(\mu) \sum_{n=0}^{N} \big(h_n^{\lambda}\big)^{-1} \big(1 - \mu^2\big)^{\lambda - 1/2} P_n^{(\lambda)}(\mu)\,\mathrm{d}\mu \times \sum_{j=1}^{(N+1)/2} g_n(\nu_j)\big[A_j \mathrm{e}^{-\tau/\nu_j} + (-1)^n B_j \mathrm{e}^{\tau/\nu_j}\big] \tag{34}$$

The orthogonality relation of ultraspherical polynomials is [18]

$$\int_{-1}^{1} \big(1 - \mu'^2\big)^{\lambda - 1/2} P_m^{\lambda}(\mu') P_n^{(\lambda)}(\mu')\,\mathrm{d}\mu' = \delta_{mn} h_n^{\lambda} \tag{35}$$

Then, with $m = 1$, $q_h(\tau)$ can be simplified to

$$q_h(\tau) = \frac{\pi}{\lambda} \sum_{j=1}^{(N+1)/2} g_1(\nu_j)\big[A_j \mathrm{e}^{-\tau/\nu_j} - B_j \mathrm{e}^{\tau/\nu_j}\big] \tag{36}$$

The linearity of the governing equations permits the construction of the solution to the general problem by superposition of elementary solutions independent of temperature. In the following section, for cases of (1) constant, (2) linearly varying, and (3) exponentially varying source terms, particular solutions to the radiative transfer equation with linearly anisotropic scattering kernel are to be presented and, applying the principle of superposition, radiative heat fluxes are to be expressed in terms of dimensionless functions.

Note that, for linearly anisotropic scattering, Özişik and Siewert [19] have obtained several particular solutions of the one-speed transport equation, which is equivalent to the radiative transfer equation, and the expressions given in that study can be used to treat problems with more general source terms.

4.1. Constant source term

For the constant source term $S(\tau) = (1 - \omega)/\pi$, it is an easy task to show that a particular solution to the radiative transfer equation is

$$I_p(\tau, \mu) = \frac{1}{\pi} \tag{37}$$

Moreover, for non-conservative media, when $S(\tau)$ is a constant, it can be shown that the net radiative heat flux is [3]

$$q(\tau) = \sigma\big[T_0^4 Q_0(\tau) + T_1^4 Q_1(\tau) - T_2^4 Q_1^*(\tau_0 - \tau)\big] \tag{38}$$

where Q_i (for $i = 0, 1$) are dimensionless heat flux functions, Q_1^* is the dimensionless heat flux function obtained using the principle of reflection, T_1 and T_2 are the temperatures at the boundary surfaces $\tau = 0$ and $\tau = \tau_0$, respectively, and T_0 is the uniform temperature in the medium. The dimensionless functions $Q_i(\tau)$ are defined as

$$Q_i(\tau) = 2\pi \int_{-1}^{1} \mu \psi_i(\tau, \mu)\,\mathrm{d}\mu, \quad i = 0 \text{ or } 1 \tag{39}$$

where $\psi_i(\tau, \mu)$ are dimensionless functions that satisfy the following system:

$$\mu \frac{\partial \psi_i(\tau, \mu)}{\partial \tau} + \psi_i(\tau, \mu) - \frac{\omega}{2} \int_{-1}^{1} p(\mu, \mu')\psi_i(\tau, \mu')\,\mathrm{d}\mu' = \frac{1 - \omega}{\pi} \delta_{0i}, \quad i = 0 \text{ or } 1 \tag{40}$$

$$\psi_i(0, \mu) = \frac{1}{\pi} \varepsilon_1 \delta_{1i} + b_1 \psi_i(0, -\mu) + d_1 \int_{0}^{1} \psi_i(0, -\mu')\mu'\,\mathrm{d}\mu', \quad \mu \in [0, 1] \tag{41}$$

$$\psi_i(\tau_0,-\mu)=b_2\psi_i(\tau_0,\mu)+d_2\int_0^1\psi_i(\tau_0,\mu')\mu'\,\mathrm{d}\mu'$$

$$\mu\in[0,1] \qquad (42)$$

where δ_{ij} is the Kronecker delta function.

The function $Q_1^*(\tau_0-\tau)$ is obtained from the solution of the system defining $Q_1(\tau)$ by interchanging radiative properties at the boundary surfaces 1 and 2.

For conservative media ($\omega=1$) with opaque boundaries, Eq. (38) simplifies to

$$q(\tau)=\sigma\left[T_1^4-T_2^4\right]Q_1 \qquad (43)$$

where Q_1 is a constant obtained as the solution of the equations defining $Q_1(\tau)$.

4.2. Linearly varying source term

For the linearly varying source term $S(\tau)=(1/\pi)(1-\omega)(1-\tau/\tau_0)$, a particular solution to the transfer equation is [17]

$$I_p(\tau,\mu)=\frac{1}{\pi}\left[1-\frac{\tau-\mu(1-\omega B/3)^{-1}}{\tau_0}\right] \qquad (44)$$

When the fourth power of the temperature of the medium varies linearly from the surface temperature T_1 at $\tau=0$ to the surface temperature T_2 at $\tau=\tau_0$, it can be shown that the net radiative heat flux is [3]

$$q(\tau)=\sigma\left[T_1^4\overline{Q}_1(\tau)-T_2^4\overline{Q}_1^*(\tau_0-\tau)\right] \qquad (45)$$

The dimensionless function $\overline{Q}_1(\tau)$ is defined as

$$\overline{Q}_1(\tau)=2\pi\int_{-1}^1\mu\overline{\psi}_1(\tau,\mu)\,\mathrm{d}\mu \qquad (46)$$

where the function $\overline{\psi}_1(\tau,\mu)$ satisfies the following system:

$$\mu\frac{\partial\overline{\psi}_1(\tau,\mu)}{\partial\tau}+\overline{\psi}_1(\tau,\mu)-\frac{\omega}{2}\int_{-1}^1 p(\mu,\mu')\overline{\psi}_1(\tau,\mu)\,\mathrm{d}\mu' =\frac{1-\omega}{\pi}\left(1-\frac{\tau}{\tau_0}\right) \qquad (47)$$

$$\overline{\psi}_1(0,\mu)=\frac{1}{\pi}\varepsilon_1+b_1\overline{\psi}_1(0,-\mu)+d_1\int_0^1\overline{\psi}_1(0,-\mu')\mu'\,\mathrm{d}\mu'$$

$$\mu\in[0,1] \qquad (48)$$

$$\overline{\psi}_1(\tau_0,-\mu)=b_2\overline{\psi}_1(\tau_0,\mu)+d_2\int_0^1\overline{\psi}_1(\tau_0,\mu')\mu'\,\mathrm{d}\mu'$$

$$\mu\in[0,1] \qquad (49)$$

The function $\overline{Q}_1^*(\tau_0-\tau)$ is obtained from the solution of the system defining $\overline{Q}_1(\tau)$ by interchanging radiative properties at the boundary surfaces 1 and 2.

4.3. Exponentially varying source term

The methodology followed for the problem with the linearly varying source term, in Section 4.2, can be generalized to any problem provided that the source term has a specified value on the boundary $\tau=0$ and vanishes on the boundary $\tau=\tau_0$. In this section, this generalization is illustrated for an exponentially varying source term. An exponentially varying source function that has a specified value on the boundary $\tau=0$ and vanishes on the boundary $\tau=\tau_0$ is in the form

$$S(\tau)=\frac{1-\omega}{\pi}\left(\frac{\mathrm{e}^{-\tau_0/\eta}-\mathrm{e}^{-\tau/\eta}}{\mathrm{e}^{-\tau_0/\eta}-1}\right),\quad |\eta|>1 \qquad (50)$$

where η is an adjustable parameter describing how fast the source term varies.

A particular solution can be constructed as

$$I_p(\tau,\mu)=\frac{1}{\pi}\frac{\mathrm{e}^{-\tau_0/\eta}-(1-\omega)F(\eta,\mu)\mathrm{e}^{-\tau/\eta}}{\mathrm{e}^{-\tau_0/\eta}-1} \qquad (51)$$

where

$$F(\eta,\mu)=\frac{\eta}{\eta-\mu}\left[1+\frac{\omega}{2}K(\eta)+\frac{\omega}{2}B\mu L(\eta)\right] \qquad (52)$$

$$K(\eta)=2\frac{\frac{A(\eta)}{\eta}+1+\frac{\omega BA^2(\eta)}{1-\omega BA(\eta)}}{1-\omega\left(\frac{A(\eta)}{\eta}+1\right)-\frac{\omega^2BA^2(\eta)}{1-\omega BA(\eta)}} \qquad (53)$$

$$L(\eta)=A(\eta)\frac{2+\omega K(\eta)}{1-\omega BA(\eta)} \qquad (54)$$

$$A(\eta)=\eta\left(-1+\eta\tanh^{-1}\frac{1}{\eta}\right) \qquad (55)$$

When the fourth power of the temperature of the medium varies exponentially as in Eq. (50), the net radiative heat flux is obtained as

$$q(\tau)=\sigma\left[T_1^4\widehat{Q}_1(\tau)-T_2^4\widehat{Q}_1^*(\tau_0-\tau)\right] \qquad (56)$$

The dimensionless function $\widehat{Q}_1(\tau)$ is defined as

$$\widehat{Q}_1(\tau)=2\pi\int_{-1}^1\mu\widehat{\psi}_1(\tau,\mu)\,\mathrm{d}\mu \qquad (57)$$

where the function $\widehat{\psi}_1(\tau,\mu)$ satisfies the following system:

$$\mu\frac{\partial\widehat{\psi}_1(\tau,\mu)}{\partial\tau}+\widehat{\psi}_1(\tau,\mu)-\frac{\omega}{2}\int_{-1}^1 p(\mu,\mu')\widehat{\psi}_1(\tau,\mu)\,\mathrm{d}\mu' =\frac{1-\omega}{\pi}\left(\frac{\mathrm{e}^{-\tau_0/\eta}-\mathrm{e}^{-\tau/\eta}}{\mathrm{e}^{-\tau_0/\eta}-1}\right) \qquad (58)$$

$$\widehat{\psi}_1(0,\mu)=\frac{1}{\pi}\varepsilon_1+b_1\widehat{\psi}_1(0,-\mu)+d_1\int_0^1\widehat{\psi}_1(0,-\mu')\mu'\,\mathrm{d}\mu'$$

$$\mu\in[0,1] \qquad (59)$$

$$\widehat{\psi}_1(\tau_0,-\mu)=b_2\widehat{\psi}_1(\tau_0,\mu)+d_2\int_0^1\widehat{\psi}_1(\tau_0,\mu')\mu'\,\mathrm{d}\mu'$$

$$\mu\in[0,1] \qquad (60)$$

Table 1
The heat flux functions $Q_0(0)$, $Q_1(0)$ and $Q_1^*(\tau_0)$ for non-conservative media at constant temperature for various $P_N^{(\lambda)}$ approximations ($N = 21$)

$\tau = 0$		$\tau = \tau_0$		$\tau_0 = 0.1$		$\tau_0 = 1.0$		$\tau_0 = 10.0$	
ε_1	ρ_1^d	ε_2	ρ_2^d	$\omega = 0$	$\omega = 0.5$	$\omega = 0$	$\omega = 0.5$	$\omega = 0$	$\omega = 0.5$
				$\lambda = -0.4999$					
				$-Q_0(0)$					
1.0	0.0	0.0	1.0	0.308706	0.174231	0.952688	0.757588	1.00078	0.853845
1.0	0.0	0.5	0.5	0.238652	0.132044	0.867077	0.651423	1.00077	0.853823
1.0	0.0	1.0	0.0	0.168546	0.091464	0.781399	0.559460	1.00077	0.853804
				$Q_1(0)$					
1.0	0.0	0.0	1.0	0.308706	0.174231	0.952688	0.757588	1.00078	0.853845
1.0	0.0	0.5	0.5	0.654594	0.575715	0.976723	0.815802	1.00078	0.853845
1.0	0.0	1.0	0.0	1.000740	0.961905	1.000780	0.866229	1.00078	0.853845
				$Q_1^*(\tau_0)$					
1.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
1.0	0.0	0.5	0.5	0.415942	0.443671	0.109646	0.164379	0.0	0.0
1.0	0.0	1.0	0.0	0.832191	0.870441	0.219378	0.306769	0.0	0.0
				$\lambda = 0.0$ (Chebyshev polynomial of the first kind-T_N approximation)					
				$-Q_0(0)$					
1.0	0.0	0.0	1.0	0.308713	0.174254	0.952642	0.757547	1.00076	0.853838
1.0	0.0	0.5	0.5	0.238659	0.132064	0.867044	0.651401	1.00076	0.853816
1.0	0.0	1.0	0.0	0.168556	0.091481	0.781383	0.559455	1.00075	0.853798
				$Q_1(0)$					
1.0	0.0	0.0	1.0	0.308647	0.174279	0.952516	0.757477	1.00059	0.853695
1.0	0.0	0.5	0.5	0.654478	0.575660	0.976544	0.815669	1.00059	0.853695
1.0	0.0	1.0	0.0	1.000550	0.961743	1.000590	0.866077	1.00059	0.853695
				$Q_1^*(\tau_0)$					
1.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
1.0	0.0	0.5	0.5	0.415903	0.443612	0.109627	0.164349	0.0	0.0
1.0	0.0	1.0	0.0	0.832094	0.870316	0.219335	0.306711	0.0	0.0
				$\lambda = 1/2$ (spherical harmonics-P_N approximation)					
				$-Q_0(0)$					
1.0	0.0	0.0	1.0	0.308543	0.174179	0.952615	0.757553	1.00071	0.853810
1.0	0.0	0.5	0.5	0.238520	0.132003	0.867003	0.651389	1.00070	0.853788
1.0	0.0	1.0	0.0	0.168449	0.091436	0.781330	0.559430	1.00070	0.853769
				$Q_1(0)$					
1.0	0.0	0.0	1.0	0.308543	0.174179	0.952615	0.757553	1.00071	0.853810
1.0	0.0	0.5	0.5	0.654491	0.575680	0.976653	0.815768	1.00071	0.853810
1.0	0.0	1.0	0.0	1.000670	0.961872	1.000710	0.866193	1.00071	0.853810
				$Q_1^*(\tau_0)$					
1.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
1.0	0.0	0.5	0.5	0.415971	0.443676	0.109650	0.164379	0.0	0.0
1.0	0.0	1.0	0.0	0.832222	0.870436	0.219378	0.306763	0.0	0.0
				$\lambda = 1.0$ (Chebyshev polynomial of second kind-U_N approximation)					
				$-Q_0(0)$					
1.0	0.0	0.0	1.0	0.309265	0.174635	0.952752	0.757666	1.00083	0.853905
1.0	0.0	0.5	0.5	0.239103	0.132357	0.867147	0.651503	1.00083	0.853883
1.0	0.0	1.0	0.0	0.168881	0.091687	0.781471	0.559535	1.00082	0.853865
				$Q_1(0)$					
1.0	0.0	0.0	1.0	0.309265	0.174635	0.952752	0.757666	1.00083	0.853905
1.0	0.0	0.5	0.5	0.654908	0.575927	0.976782	0.815867	1.00083	0.853905
1.0	0.0	1.0	0.0	1.000840	0.961950	1.000830	0.866287	1.00083	0.853905
				$Q_1^*(\tau_0)$					
1.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
1.0	0.0	0.5	0.5	0.415806	0.443570	0.109635	0.164365	0.0	0.0
1.0	0.0	1.0	0.0	0.831962	0.870263	0.219360	0.306752	0.0	0.0

Table 1 (*Continued*)

$\tau=0$		$\tau=\tau_0$		$\tau_0=0.1$		$\tau_0=1.0$		$\tau_0=10.0$	
ε_1	ρ_1^d	ε_2	ρ_2^d	$\omega=0$	$\omega=0.5$	$\omega=0$	$\omega=0.5$	$\omega=0$	$\omega=0.5$
					$\lambda=2.0$				
					$-Q_0(0)$				
1.0	0.0	0.0	1.0	0.325137	0.184269	0.956111	0.760236	1.003890	0.856110
1.0	0.0	0.5	0.5	0.251949	0.139839	0.870662	0.654079	1.003890	0.856088
1.0	0.0	1.0	0.0	0.178406	0.097018	0.784880	0.561886	1.003890	0.856069
					$Q_1(0)$				
1.0	0.0	0.0	1.0	0.325137	0.184269	0.956111	0.760236	1.003890	0.856110
1.0	0.0	0.5	0.5	0.664171	0.581219	0.979955	0.818157	1.003890	0.856110
1.0	0.0	1.0	0.0	1.004850	0.963798	1.003890	0.868458	1.003890	0.856110
					$Q_1^*(\tau_0)$				
1.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
1.0	0.0	0.5	0.5	0.412222	0.441380	0.109293	0.164077	0.0	0.0
1.0	0.0	1.0	0.0	0.826443	0.866781	0.219012	0.306572	0.0	0.0

Table 2
Comparison of the heat flux function Q_1 obtained from different first order ultraspherical polynomials approximation $P_1^{(\lambda)}$ for conservative media at constant temperature

$\tau=0$		$\tau=\tau_0$		λ	$\tau_0=0.1$	Abs. error (%)	$\tau_0=1.0$	Abs. error (%)	$\tau_0=10.0$	Abs. error (%)
ε_1	ρ_1^d	ε_2	ρ_2^d							
1.0	0.0	0.0	1.0	−0.4999	0.0	0.00	0.0	0.00	0.0	0.00
				0 (T_N)	0.0	0.00	0.0	0.00	0.0	0.00
				1/2 (P_N)	0.0	0.00	0.0	0.00	0.0	0.00
				1 (U_N)	0.0	0.00	0.0	0.00	0.0	0.00
				2	0.0	0.00	0.0	0.00	0.0	0.00
				5	0.0	0.00	0.0	0.00	0.0	0.00
1.0	0.0	0.5	0.5	−0.4999	0.487804	2.05	0.399995	12.30	0.1428510	36.65
				0 (T_N)	0.484576	1.38	0.379275	6.48	0.1195300	14.34
				1/2 (P_N)	0.481928	0.82	0.363636	2.09	0.1052630	0.69
				1 (U_N)	0.479643	0.34	0.351022	1.45	0.0953447	8.80
				2	0.475769	0.47	0.331280	7.00	0.0820619	21.50
				5	0.467049	2.29	0.293166	17.70	0.0620719	40.62
1.0	0.0	1.0	0.0	−0.4999	0.952378	4.00	0.666653	20.46	0.1666580	42.75
				0 (T_N)	0.940013	2.66	0.610928	10.40	0.1357360	16.26
				1/2 (P_N)	0.930233	1.59	0.571429	3.26	0.1176470	0.77
				1 (U_N)	0.921759	0.66	0.540884	2.26	0.1053930	9.73
				2	0.907557	0.89	0.495395	10.48	0.0893981	23.43
				5	0.876345	4.30	0.414759	25.05	0.0661798	43.31

The function $\widehat{Q}_1^*(\tau_0-\tau)$ is obtained from the solution of the system defining $\widehat{Q}_1(\tau)$ by interchanging radiative properties at the boundary surfaces 1 and 2.

5. Numerical results and assessments

In an absorbing, emitting, non-isothermal, gray medium between two parallel reflecting boundaries, numerical results of $P_N^{(\lambda)}$ solutions are to be presented for the constant, linearly varying, and exponentially varying source terms. For the constant and linearly varying source terms, results from the $P_N^{(\lambda)}$ approximation are assessed by comparing to the benchmark quality results of Beach et al. [16] and, when proper, to the spherical harmonics (P_N) results of Atalay [17]. For the exponentially varying source term, comparisons are made with results obtained from the CFD code FLUENT.

5.1. Constant source term

For the constant source term and isotropic scattering, the system defined in Eqs. (35)–(37) is solved by the $P_N^{(\lambda)}$ method for three different optical thicknesses ($\tau_0=0.1$, 1.0 and 10.0) and for different combinations of reflectivities and emissivities.

5.1.1. Non-conservative case

For the non-conservative case, numerical results for heat flux functions $Q_0(0)$, $Q_1(0)$ and $Q_1^*(\tau_0)$ are presented in Table 1. The code developed proceeds up to an order of approximation $N=40$ before numerical instabilities begin to be observed. As stated by the Equiconvergence Theorem of Jacobi Polynomials [18], ultraspherical polynomials are equiconvergent. For high-order approximations (around $N=35$), results from all $P_N^{(\lambda)}$ approximations under consideration are in perfect agree-

Table 3
The heat flux functions $\overline{Q}_1(0)$ and $\overline{Q}_1^*(\tau_0)$ for non-conservative media with a linear fourth-power of temperature ($N = 21$)

$\tau = 0$		$\tau = \tau_0$		$\tau_0 = 0.1$		$\tau_0 = 1.0$		$\tau_0 = 10.0$	
ε_1	ρ_1^d	ε_2	ρ_2^d	$\omega = 0$	$\omega = 0.5$	$\omega = 0$	$\omega = 0.5$	$\omega = 0$	$\omega = 0.5$
$\lambda = -0.4999$									
$\overline{Q}_1(0)$									
1.0	0.0	0.0	1.0	0.153615	0.086905	0.386118	0.335215	0.066663	0.075772
1.0	0.0	0.5	0.5	0.532708	0.508461	0.440298	0.433707	0.066664	0.075774
1.0	0.0	1.0	0.0	0.912081	0.913957	0.494521	0.519024	0.066664	0.075775
$\overline{Q}_1^*(\tau_0)$									
1.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
1.0	0.0	0.5	0.5	0.455873	0.465852	0.247164	0.278114	0.033319	0.040875
1.0	0.0	1.0	0.0	0.912081	0.913957	0.494521	0.519024	0.066664	0.075775
$\lambda = 0.0$ (Chebyshev polynomial of first kind-T_N approximation)									
$\overline{Q}_1(0)$									
1.0	0.0	0.0	1.0	0.153572	0.086954	0.385960	0.335114	0.066490	0.075627
1.0	0.0	0.5	0.5	0.532657	0.508467	0.440127	0.433576	0.066491	0.075629
1.0	0.0	1.0	0.0	0.912006	0.913915	0.494334	0.518865	0.066491	0.075631
$\overline{Q}_1^*(\tau_0)$									
1.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
1.0	0.0	0.5	0.5	0.455961	0.465930	0.247177	0.278118	0.033324	0.040880
1.0	0.0	1.0	0.0	0.912107	0.913970	0.494463	0.518955	0.066662	0.075774
$\lambda = 1/2$ (spherical harmonics P_N-approximation)									
$\overline{Q}_1(0)$									
1.0	0.0	0.0	1.0	0.153532	0.086878	0.386113	0.335208	0.066664	0.075771
1.0	0.0	0.5	0.5	0.532665	0.508442	0.440299	0.433704	0.066664	0.075773
1.0	0.0	1.0	0.0	0.912053	0.913932	0.494524	0.519020	0.066664	0.075775
$\overline{Q}_1^*(\tau_0)$									
1.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
1.0	0.0	0.5	0.5	0.455873	0.465847	0.247174	0.278117	0.033320	0.040875
1.0	0.0	1.0	0.0	0.912053	0.913932	0.494524	0.51902	0.066664	0.075775
$\lambda = 1.0$ (Chebyshev polynomial of second kind U_N-approximation)									
$\overline{Q}_1(0)$									
1.0	0.0	0.0	1.0	0.154022	0.087185	0.386029	0.335171	0.066648	0.075760
1.0	0.0	0.5	0.5	0.533245	0.508978	0.440186	0.433628	0.066649	0.075762
1.0	0.0	1.0	0.0	0.912787	0.914722	0.494388	0.518920	0.066649	0.075764
$\overline{Q}_1^*(\tau_0)$									
1.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
1.0	0.0	0.5	0.5	0.456201	0.466231	0.247091	0.27805	0.0333105	0.0408672
1.0	0.0	1.0	0.0	0.912787	0.914722	0.494388	0.51892	0.0666487	0.0757639
$\lambda = 2.0$									
$\overline{Q}_1(0)$									
1.0	0.0	0.0	1.0	0.164699	0.093690	0.384375	0.334463	0.066338	0.075545
1.0	0.0	0.5	0.5	0.545251	0.519933	0.437899	0.432130	0.066338	0.075547
1.0	0.0	1.0	0.0	0.927648	0.930745	0.491631	0.516950	0.066338	0.075549
$\overline{Q}_1^*(\tau_0)$									
1.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
1.0	0.0	0.5	0.5	0.462702	0.473951	0.245338	0.276672	0.033105	0.040703
1.0	0.0	1.0	0.0	0.927648	0.930745	0.491631	0.516950	0.066338	0.075549

ment with the exact analytical results of Beach et al. [16] and the spherical-harmonics results of Atalay [17]. Therefore, it makes no sense to tabulate them. The numerical values tabulated in Table 1 are for $N = 21$, which allows to observe some qualitative aspects of ultraspherical-polynomials approximations.

It can be observed from Table 1 that, in the diffusion limit (optically thick), rate of convergence in the P_N method is better than those in other ultraspherical-polynomials approximations, as expected.

For small values of λ, uniform convergence is achieved in all the cases listed in Table 1. On the contrary, for λ greater than some value above 2, depending on the setup of the problem, convergence in the mean is observed and results highly oscillate around the mean. This highly oscillatory behavior encountered at large values of λ is primarily due to a change in the nature

Table 4
The heat flux functions $\overline{Q}_1(0)$ for non-conservative media with a purely specularly reflecting boundary at $\tau = \tau_0$ ($N = 35$)

λ	Boundary at $\tau = 0$		Boundary at $\tau = \tau_0$		$\tau_0 = 0.1$, $\omega = 0.5$	Abs. error (%)	$\tau_0 = 1.0$, $\omega = 0.5$	Abs. error (%)	$\tau_0 = 10.0$, $\omega = 0.5$	Abs. error (%)
	ε_1	ρ_1^s	ε_2	ρ_2^s	$\overline{Q}_1(0)$		$\overline{Q}_1(0)$		$\overline{Q}_1(0)$	
−0.4999	1.0	0.0	0.0	1.0	0.084521	0.025	0.326333	0.010	0.075767	0.044
	1.0	0.0	0.5	0.5	0.506660	0.012	0.428502	0.000	0.075769	0.041
0* (T_N)	1.0	0.0	0.0	1.0	0.084509	0.011	0.326228	0.022	0.075642	0.209
	1.0	0.0	0.5	0.5	0.506589	0.002	0.428385	0.027	0.075644	0.206
1/2 (P_N)	1.0	0.0	0.0	1.0	0.084526	0.030	0.326332	0.010	0.075767	0.044
	1.0	0.0	0.5	0.5	0.506664	0.013	0.428501	0.000	0.075769	0.041
1 (U_N)	1.0	0.0	0.0	1.0	0.084370	0.154	0.326352	0.016	0.075770	0.400
	1.0	0.0	0.5	0.5	0.506524	0.015	0.428525	0.006	0.075772	0.037
3/2	1.0	0.0	0.0	1.0	0.083232	1.500	0.326494	0.059	0.075789	0.014
	1.0	0.0	0.5	0.5	0.505509	0.215	0.428695	0.046	0.075791	0.011
2	1.0	0.0	0.0	1.0	0.078596	6.987	0.327100	0.245	0.075875	0.099
	1.0	0.0	0.5	0.5	0.501403	1.026	0.429421	0.215	0.075877	0.102

of the eigenspectrum of the transport operator. Small values of λ yield real eigenvalues in pairs while large values of λ produce both complex conjugate and real pairs. As an already-large value of λ is increased further, the number of complex conjugate pairs rises and oscillatory nature becomes more dominant; finally, all eigenvalue pairs become complex.

5.1.2. Conservative case

For conservative media, numerical values of the heat flux function Q_1, obtained from $P_1^{(\lambda)}$ approximation for a value of ω very very close to unity ($\omega = 0.999999999999$), are presented in Table 2. In order to better observe the differences between results of several approximations, the first-order approximation ($N = 1$) is selected. That is also the reason why some of the absolute errors appearing in Table 2 are large; it is obvious that all the error terms will die away as the order of approximation is increased. Looking at the absolute errors in Table 2, calculated by comparison to the exact values of Ref. [16], it can be stated that P_1 approximation produces by far the best results for optically thick media. For optically thin media, in which effects of anisotropies become very important, U_1 approximation is superior to the others.

5.2. Linearly varying source term

For the linearly varying source term, isotropic scattering and non-conservative media, numerical values of the heat flux functions $\overline{Q}_1(0)$ and $\overline{Q}_1^*(\tau_0)$, obtained from $P_{21}^{(\lambda)}$, are listed in Table 3. The equiconvergence is obvious again. A comparison of the listed values to the exact results of Ref. [16] shows that the $P_N^{(\lambda)}$ approximation with $\lambda \to -1/2$ is as good as the spherical harmonics (P_N) method.

To investigate effects of specular reflection, the heat flux function $\overline{Q}_1(0)$ is evaluated from $P_{35}^{(\lambda)}$ approximation for non-conservative media with a purely specular reflector at the boundary $\tau = \tau_0$. Results are presented in Table 4, together with absolute errors calculated by comparison to the exact values from Ref. [16]. As can be observed from Table 4, for optically thick media, results from all $P_{35}^{(\lambda)}$ approximations agree with the exact analytical values very well, with $P_N^{(3/2)}$ being slightly superior to the others. For optically thin media, a small value of λ produces relatively better results, with T_N yielding the smallest absolute error.

By comparing the results in Table 4 to those in Table 3, it can be stated that heat fluxes for specularly reflecting and diffuse reflecting cases are almost the same for optically thick media and differ only slightly for optically thin media.

To observe effects of linearly anisotropic scattering, $\overline{Q}_1(\tau)$ is depicted as a function of the dimensionless optical variable τ/τ_0 in Fig. 1 for each $P_{35}^{(\lambda)}$ approximation. Because of their equiconvergence, results from all $P_{35}^{(\lambda)}$ approximations are consistent in themselves and in perfect agreement with the exact results of Ref. [16]. In all $P_{35}^{(\lambda)}$ approximations, the net radiative heat flux is slightly higher for linearly isotropic scattering with $B = 0.9$ than that for isotropic scattering. The difference between heat fluxes is very small in optically thin media, small in optically thick media, and gets smaller as τ_0 is increased further. The same trend can also be observed in the analytical results of Ref. [16] and in the spherical-harmonics results of Ref. [17].

5.3. Exponentially varying source term

For the exponentially varying source term, linearly anisotropic scattering and non-conservative media, only the most common ultraspherical-polynomials ($P_N^{(1/2)} = P_N$, $P_N^{(0)} = T_N$, $P_N^{(1)} = U_N$) approximations are taken into account and assessments are made by comparing results to those from the CFD code FLUENT.

FLUENT was used to solve the energy and the radiative transfer equations in 2D domain by a second-order discretization scheme. For the radiative transfer equation, Discrete Ordinates Model, which is the recommended model for the ranges of optical thicknesses in this study, was chosen. At the constant-temperature boundaries, σT^4 was set equal to unity; and at the other two boundaries, the symmetry condition was applied.

Two user-defined functions were coded: one to implement the source term, and the other to obtain the radiative heat flux across the calculational domains since FLUENT provides radiative heat fluxes only at boundaries. Segregated solver was

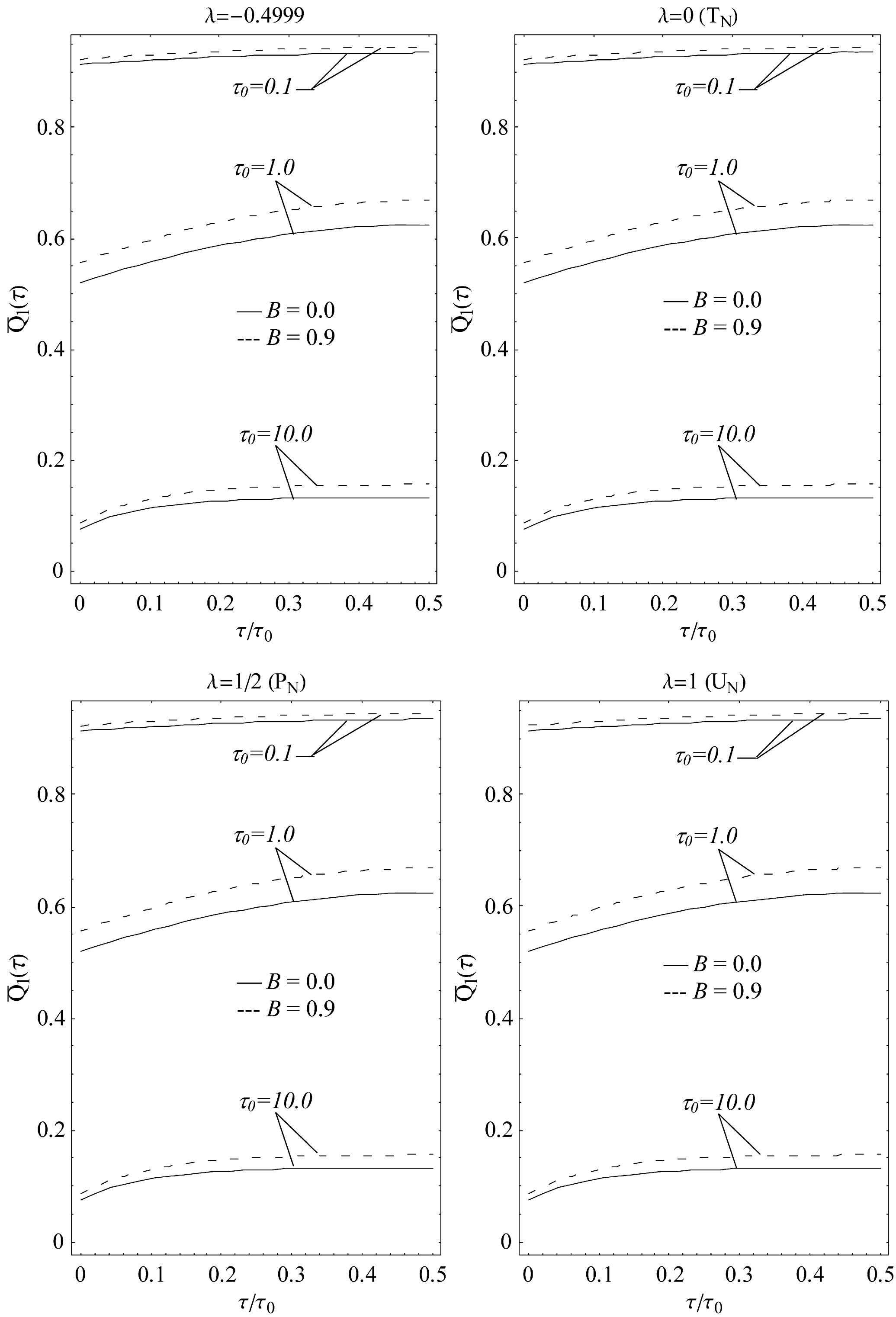
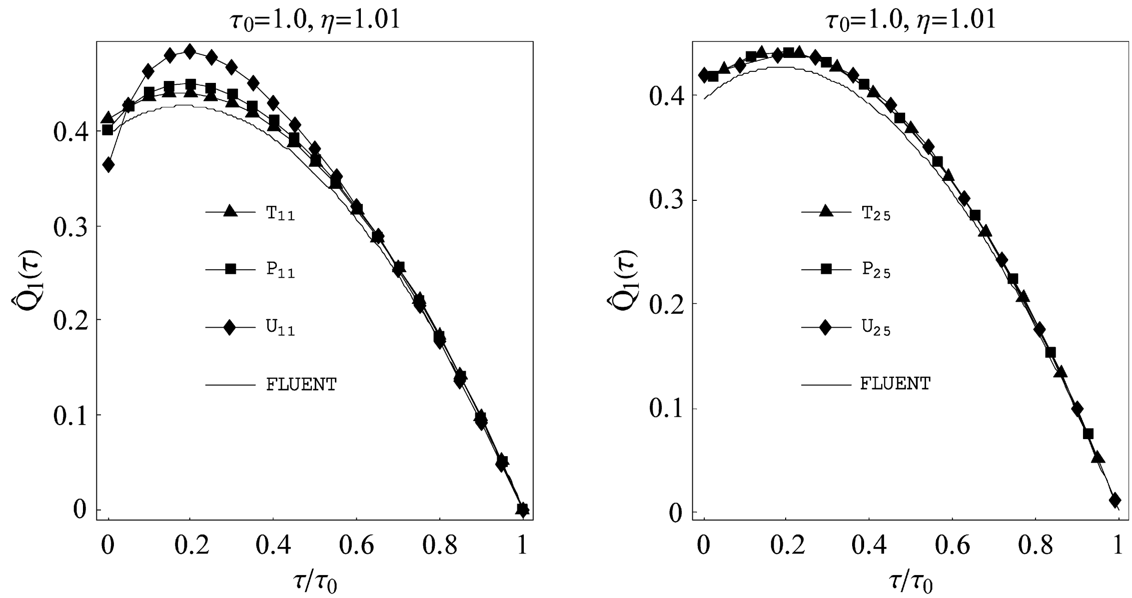


Fig. 1. The effects of linearly anisotropic scattering on the function $\overline{Q}_1(\tau)$ for $c = 0.5$ (comparison of various $P_{35}^{(\lambda)}$ approximations).

used and implicit formulation was chosen for discretization of the transfer equation. The residuals were set as 1×10^{-8} for both the energy and the radiative transfer equations. The same square quadrilateral mesh was used for both optical thicknesses $\tau_0 = 1.0$ and $\tau_0 = 10.0$. The computational domain was discretized with mesh numbers 2000 and 20 000 for $\tau_0 = 1.0$ and 10.0, respectively. Each octant of the angular space 4π at any spatial location was discretized into $N_\theta \times N_\phi$ solid angles, which yielded a total of $4N_\theta N_\phi$ directions since, in 2D calculations, only four octants are considered due to symmetry.

To determine effects of angular discretization, calculations were performed for $N_\theta = N_\phi = 2$, 4 and 8 for each optical



(a) (b)

Fig. 2. Evolution of different $P_N^{(\lambda)}$ approximation solutions by order of approximation N.

thickness. It was observed that the order of the angular discretization did not effect the results; yet, the computational time was significantly longer for a higher order of angular discretization. When $\tau_0 = 1.0$, the computational time until convergence was 40 seconds for $N_\theta = N_\phi = 2$, 1 minute 45 seconds for $N_\theta = N_\phi = 4$, and 5 minutes 45 seconds for $N_\theta = N_\phi = 8$ on a Pentium IV-2.8 GHz machine. Therefore, computations were performed only for $N_\theta = N_\phi = 2$, which took 308 and 637 iterations until convergence for $\tau_0 = 1.0$ and $\tau_0 = 10.0$, respectively. As for the $P_{25}^{(\lambda)}$ approximation, the computational time did not exceed about 3 seconds on the same machine for an accuracy of four significant figures.

For the following set of parameters ($\omega = 0.5$, $B = 0.9$, $\varepsilon_1 = 1.0$, $\varepsilon_2 = 0.0$, $\rho_1^s = 0.0$, $\rho_2^s = 0.0$, $\rho_1^d = 0.0$, $\rho_2^d = 1.0$), numerical values of the heat flux function $\widehat{Q}_1(\tau)$, obtained from T_N, P_N and U_N approximations and from FLUENT, are depicted versus τ/τ_0 in Fig. 2 for $(\tau_0, \eta) = (1.0, 1.01)$ and for $N = 11$ and 25. Observing the plots in Fig. 2, it can be concluded that the $P_N^{(\lambda)}$ approximation with small λ evolves faster to the solution. As can easily be seen in Fig. 2(b), all results from the $P_{25}^{(\lambda)}$ approximations completely overlap and are in good agreement with the FLUENT results except for a region around the maximum.

To see effect of change of the source strength, value of η (see Eq. (43)) is increased from 1.01 to 5.0, making the exponential source less varying. $\widehat{Q}_1(\tau)$ values, computed from the $P_{25}^{(\lambda)}$ approximations for $\tau_0 = 1.0$, are plotted in Fig. 3. As observed from Fig. 3, the results are equiconvergent, consistent in themselves, and comply with the FLUENT results remarkably well.

Calculations are repeated for $(\tau_0, \eta) = (10.0, 1.01)$ and $(\tau_0, \eta) = (10.0, 5.0)$, and results are plotted in Fig. 4. For optically thick media, all the $P_{25}^{(\lambda)}$ results are again consistent, equiconvergent, and in good agreement with the results of FLUENT except for a narrow region around the maximum heat flux.

6. Conclusions and discussion

In an attempt to obtain approximate solutions to the radiative transfer equation in slab geometry, all approximations using ultraspherical polynomials have been incorporated into a single formulation, named ultraspherical-polynomials approximation and denoted by $P_N^{(\lambda)}$. Each value of λ leads to a different approximation, including the spherical harmonics $[P_N = P_N^{(1/2)}]$ and Chebyshev polynomials of the first and the second kinds: $[T_N = P_N^{(0)}]$ and $[U_N = P_N^{(1)}]$, respectively.

In a plane-parallel, absorbing, emitting, non-isothermal, gray medium with linearly anisotropic scattering, numerical values of the dimensionless heat flux functions have been obtained by the $P_N^{(\lambda)}$ method. Effects of the order of approximation, optical thickness, specular reflection, anisotropic scattering, and change of the source term on results have been analyzed for different pre-selected values of λ. All the $P_N^{(\lambda)}$ results are consistent in themselves, equiconvergent, and in good agreement with the comparable data.

The unique $P_N^{(\lambda)}$ formulation allows to easily investigate some qualitative mathematical aspects (related to convergence characteristics) of various ultraspherical-polynomials approximations. In general, while small values of λ result in uniform convergence, large values of λ (above a value near 2) produce results oscillating around the mean, which retards convergency. This behavior can be attributed to the fact that the eigenspectrum space of the transport operator gradually switches from the real domain to the complex domain. The oscillatory nature tends to become more pronounced as a large value of λ is increased further.

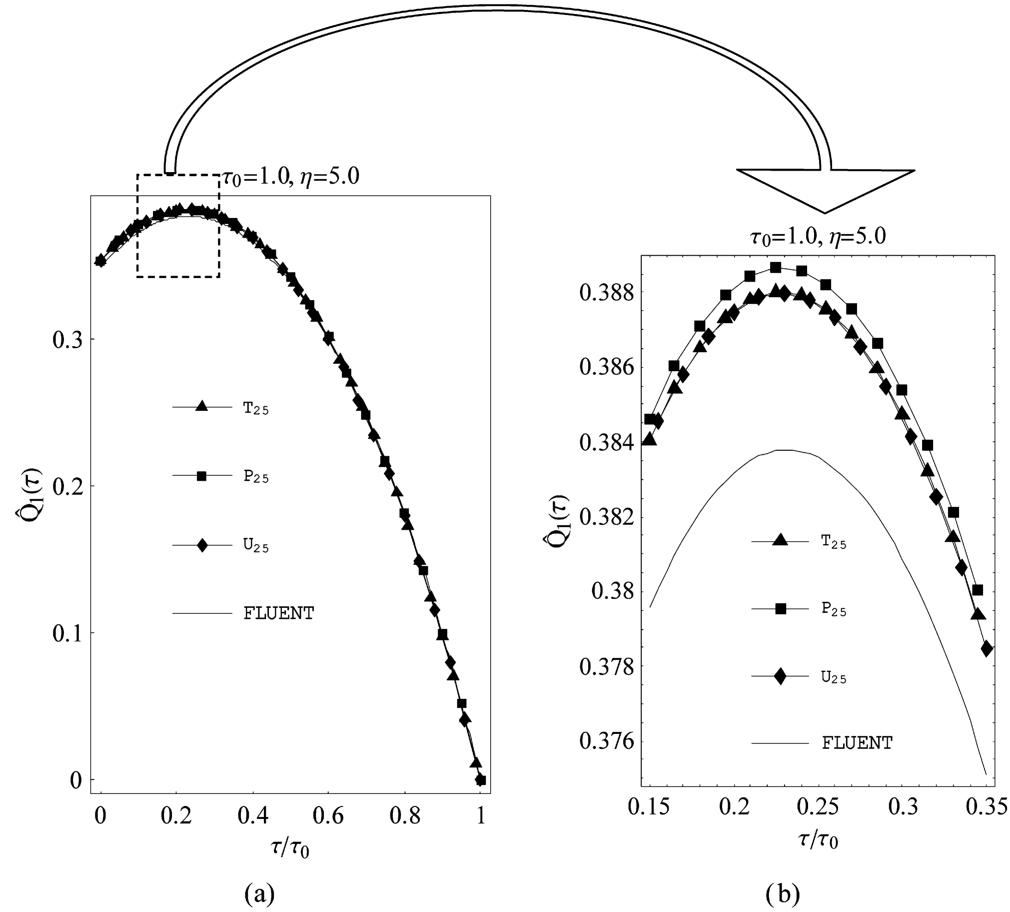


Fig. 3. The heat flux function $\widehat{Q}_1(\tau)$ obtained from the most familiar $P_N^{(\lambda)}$ approximations for the exponentially varying source, and comparison with the FLUENT results.

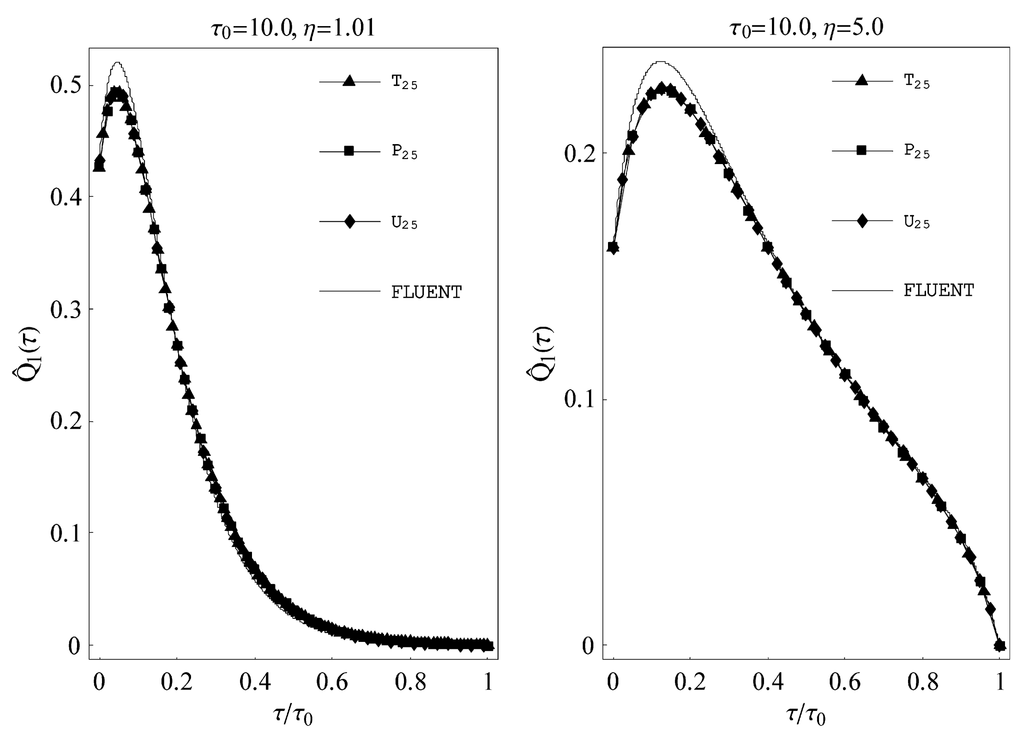


Fig. 4. Distribution of the heat flux functions $\widehat{Q}_1(\tau)$ in an optically thick medium for the exponentially varying source, and comparison with the FLUENT results.

It can be concluded that the $P_N^{(\lambda)}$ method yields satisfactory results and provides a good tool to make comparative assessments and to investigate some qualitative aspects of various ultraspherical-polynomials approximations.

Further analysis regarding the application of the more general case of Jacobi polynomials to the solution of the transfer equation is going to be the subject of another article under preparation. Application of the $P_N^{(\lambda)}$ method to the other coordinate systems, which is quite complicated except for the spherical-harmonics method, awaits to be focused on later.

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